Math 565: Functional Analysis
Lecture 3

Theorem. For $1 \le p < \infty$, $L^p(\mu)$ is a Banach space.

Proof. Using the absolutely convergent series criterion, we need to take a series $\sum_n f_n$ which converges absolutely, i.e. $\sum_n \|f_n\|_p < \infty$, and prove that $\sum_n f_n$ exists and is in L^p.

Put $g := \sum_{n \in \mathbb{N}} |f_n|$. Then applying MCT to $\left(\sum_{n<N} |f_n|\right)^p \nearrow g^p$ and using Δ-inequality gives:

$$\|g\|_p = \lim_{N\to\infty} \left\|\sum_{n<N} f_n\right\|_p \le \lim_{N\to\infty} \sum_{n<N} \|f_n\|_p = \sum_{n\in\mathbb{N}} \|f_n\|_p < \infty.$$

Thus $g \in L^p$ and in particular $g < \infty$ a.e., which means that $f := \sum_{n\in\mathbb{N}} f_n$ converges a.e. Finally, $|f - \sum_{n<N} f_n|^p \le (|f| + \sum_{n<N} |f_n|)^p \le (g+g)^p = 2^p \cdot g^p$ so DCT gives:

$\|f - \sum_{n<N} f_n\|_p \to 0$ and $\|f\|_p = \lim_{N\to\infty} \|\sum_{n<N} f_n\|_p \le \|g\|_p < \infty$. $\square$

Theorem. Simple functions are dense in L^p, $1 \le p < \infty$.

Proof. Writing a function $f \in L^p$ as $f = (f_+ - f_-) + i(f_{+i} - f_{-i})$ reduces to proving that every non-negative $f \in L^p$ is a limit (in L^p) of simple functions. But we know that there is an increasing sequence (f_n) of simple functions such that $f_n \nearrow f$ pointwise, in particular, $|f_n|^p \le |f|^p$ so $\|f_n\|_p \le \|f\|_p < \infty$ hence $f_n \in L^p$. Also, $|f - f_n|^p \le (|f| + |f_n|)^p \le 2^p |f|^p$ so by DCT, $\|f - f_n\|_p \to 0$. $\square$

Call a measure space $(X, \mathcal{M}, \mu)$ *essentiall countaby generated* if there is a ctbl $\mathcal{I} \subseteq \mathcal{M}$ such that the σ-algebra $\langle \mathcal{I} \rangle_\sigma$ generated by $\mathcal{I}$ is essentially $\mathcal{M}$, i.e. for each $M \in \mathcal{M}$ there is $B \in \langle \mathcal{I} \rangle_\sigma$ with $M =_\mu B$ (i.e. $M \Delta B$ is null)

Theorem. For $1 \le p < \infty$ and essentially ctbly generated σ-finite measure space $(X, \mathcal{M}, \mu)$, the space $L^p(X, \mathcal{M}, \mu)$ is separable.

Proof. Let $\mathcal{I} \subseteq \mathcal{M}$ be a ctbl set essentially generating $\mathcal{M}$, and let $\mathcal{A} := \langle \mathcal{I} \rangle$ be the algebra generated by $\mathcal{I}$, so $\mathcal{A}$ is ctbl.

Claim. For each $M \in \mathcal{M}$ of finite measure and $\varepsilon > 0$ there is $A \in \mathcal{A}$ with $\mu(M \Delta A) \leq \varepsilon$.

Pf of Claim. Because $M =_\mu B$ for some $B \in \langle \mathcal{I} \rangle_\sigma$, we may assume $M \in \langle \mathcal{I} \rangle_\sigma$. Since μ is σ-finite, the uniqueness part of Carathéodory's extension thm gives $\mu = (\mu|_{\mathcal{A}})^*$, so $\mu(M) = \inf\{\sum_{n \in \mathbb{N}} \mu(A_n) : M \subseteq \bigcup_{n \in \mathbb{N}} A_n, A_n \in \mathcal{A}\}$. Thus there are $A_n \in \mathcal{A}$ such that $M \subseteq \bigcup_{n \in \mathbb{N}} A_n$ and $\mu(M) \approx_{\varepsilon/2} \sum_{n \in \mathbb{N}} \mu(A_n)$. In particular, $\sum_{n \in \mathbb{N}} \mu(A_n) \leq \mu(M) + \varepsilon/2 < \infty$, so there is $N \in \mathbb{N}$ such that $\sum_{n \geq N} \mu(A_n) \leq \varepsilon/2$. Putting $A := \bigsqcup_{n < N} A_n$, we get:

$$\mu(M \Delta A) \leq \mu\Big(M \Delta \bigsqcup_{n \in \mathbb{N}} A\Big) + \mu\Big(\bigsqcup_{n \geq N} A_n\Big) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$ (Claim)

Let $\mathcal{Q}$ denote the complex-rational (i.e. $\mathbb{Q} + i\mathbb{Q}$) linear combinations of indicator functions of the sets in $\mathcal{A}$ of finite measure, so $\mathcal{Q}$ is ctbl. To show that $\mathcal{Q}$ is dense in L^p it suffices to approximate indicator functions $\mathbb{1}_M \in L^p$ with indicator functions of sets in $\mathcal{A}$ of finite measure (so those in L^p as well). By the claim, for each $M \in \mathcal{M}$ with $\mu(M) < \infty$ and $\varepsilon > 0$ there is $A \in \mathcal{A}$ with $\mu(M \Delta A) \leq \varepsilon^p$. But then

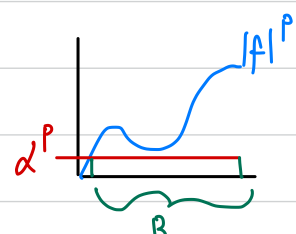
$$\int |\mathbb{1}_M - \mathbb{1}_A|^p \, d\mu = \int |\mathbb{1}_M - \mathbb{1}_A| \, d\mu = \mu(M \Delta A) \leq \varepsilon^p \text{ so } \|\mathbb{1}_M - \mathbb{1}_A\|_p \leq \varepsilon.$$

$\square$

Examples. $L^p(\mathbb{R}^d, \lambda)$, $L^p(2^{\mathbb{N}}, \text{Bernoulli}(\alpha))$, $L^p(\mathbb{N}, \text{counting measure})$ are separable for all $1 \leq p < \infty$.

Chebyshev's inequality. For $0 < p < \infty$, $f \in L^p$, and $\alpha > 0$,

$$\mu(|f| \geq \alpha) \leq \left(\frac{\|f\|_p}{\alpha}\right)^p.$$



Proof. The monotonicity of $t \mapsto t^p$ gives $B := \{|f| \geq \alpha\} = \{|f|^p \geq \alpha^p\}$, so

$$\mu(B)\cdot\alpha^p \leq \int_B \alpha^p \, d\mu \leq \int_B |f|^p \, d\mu \leq \|f\|_p^p, \text{ hence } \mu(B) \leq \frac{\|f\|_p^p}{\alpha^p}.$$ $\square$

L^∞ space.

Let (X,μ) be a measure space. Define $L^\infty(X,\mu)$ as the set (mod null) of μ-measurable functions which are bdd on a μ-conull set, i.e. $f \in L^\infty \iff \exists X' \subseteq X$ conull such that $f|_{X'}$ is bdd. This is a vector space (finite unions of null sets are null) and we define

$$\|f\|_\infty := \inf\{M \geq 0 : |f| \leq M \text{ a.e.}\}$$

In fact, this inf is a min because ctbl intersection of conull sets is conull. (Indeed, let $M_n \searrow \|f\|_\infty$ then $\{|f| \leq \|f\|_\infty\} = \bigcap_{n\in\mathbb{N}} \{|f| \leq M_n\}$ is conull.)

The value $\|f\|_\infty$ is called the essential supremum of f.

Obs. If $f \in L^\infty(X,\mu)$ then $\exists \tilde{f} = f$ a.e. such that $\|f\|_\infty = \|\tilde{f}\|_u$.

Prop. (a) $\|\cdot\|_\infty$ is a norm on $L^\infty(X,\mu)$.

(b) $L^\infty(X,\mu)$ is a Banach space.

(c) Simple functions are dense in $L^\infty(X,\mu)$.

Proof. Choosing actually bounded representatives reduces proving (a)-(c) for $B(X)$, the space of bdd functions. Thus (a) and (b) follow from the fact that $\|\cdot\|_u$ is a norm on $B(X)$ making it a Banach space. As for (c), assuming $f \in L^\infty$ is bdd, we may assume $f \geq 0$ (by writing $f = (f_+ - f_-) + i(f_{+i} - f_{-i})$). But then we know that $\exists (f_n)$ of simple functions s.t. $f_n \nearrow f$ and the convergence is uniform because f is bdd. $\square$

Notation. For $0 < p \leq \infty$ and a set X, denote $\ell^p(X) := L^p(X, \mathcal{P}(X), \text{counting measure})$.

Remark. For $d = \{0,1,\dots,d-1\}$, $\ell^p(d) = (\mathbb{C}^d, \|\cdot\|_p)$.

Because $\mathcal{P}(\mathbb{N}) = \langle \{n\} : n \in \mathbb{N} \rangle_\sigma$, $\ell^p(\mathbb{N})$ is separable for $1 \le p < \infty$. However:

Prop. $\ell^\infty(\mathbb{N})$ is not separable, neither are $L^\infty(\mathbb{R}^d, \lambda)$ and $L^\infty(2^{\mathbb{N}}, \text{Bernoulli}(\alpha))$.
In fact, $L^\infty(X, \mathcal{M}, \mu)$ is separable $\Leftrightarrow$ $(X, \mathcal{M}, \mu)$ is purely atomic with finite many disjoint atoms.

Proof. To convey the idea, it suffices to prove that $\ell^\infty(\mathbb{N})$ is not separable. The proof of the general statement is left as an exercise.

Note that for distinct $A, B \subseteq \mathbb{N}$, we have $\|\mathbb{1}_A - \mathbb{1}_B\|_\infty = 1$, so the balls $B_{\frac{1}{2}}(\mathbb{1}_A)$ of radius $1/2$ about $\mathbb{1}_A$ are pairwise disjoint for distinct $A \subseteq \mathbb{N}$. Every dense set $D \subseteq \ell^\infty(\mathbb{N})$ must meet each of these balls, so $|D| \geq |\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$. $\square$

Multiplicative properties of L^p spaces.

Let $f \in L^p$ and $g \in L^q$, what can we say about $f \cdot g$?

Prop. Geometric average $\le$ algebraic, i.e. $\forall\, a, b > 0$ and $\alpha \in (0,1)$, we have
$$a^\alpha \cdot b^{(1-\alpha)} \le \alpha \cdot a + (1-\alpha) b.$$

Proof. This follows from the convexity of exponentiation $t \mapsto e^t$. Indeed,
$$a^\alpha \cdot b^{(1-\alpha)} = (e^A)^\alpha \cdot (e^B)^{(1-\alpha)} = e^{\alpha A + (1-\alpha) B} \le \alpha \cdot e^A + (1-\alpha) e^B = \alpha a + (1-\alpha) b. \quad \square$$

Hölder's inequality. Let $0 < p, q \le \infty$, $f \in L^p$ and $g \in L^q$. Then $fg \in L^r$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ (treating $\frac{1}{\infty}$ as 0). In fact,
$$\|fg\|_r \le \|f\|_p \cdot \|g\|_q. \quad (+)$$
In particular, when p and q are conjugate exponent, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, we have
$$\|fg\|_1 \le \|f\|_p \|g\|_q. \quad (*)$$

Proof. Case $p = q = \infty$. Then $r = \infty$ and $\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty$ holds because

$$\sup_{x\in X}\left(|f(x)|\cdot|g(x)|\right) \le \sup_{x\in X}|f(x)|\cdot\sup_{x\in X}|g(x)|.$$

<u>Case $p<q=\infty$.</u> Then $r=p$ and $\|fg\|_p^p = \int |f|^p|g|^p\,d\mu \le \|g\|_\infty^p \cdot \int |f|^p\,d\mu = \|f\|_p^p\|g\|_\infty^p$.

<u>Case $p,q<\infty$.</u> (†) follows from $(*)$ applied to $|f|^r$, $|g|^r$ and $\frac{p}{r}$, $\frac{q}{p}$, so it suffices to prove $(*)$. If $\|f\|_p=0$ or $\|g\|_q=0$ then $f=0$ or $g=0$ and $(*)$ follows trivially. Thus assume $\|f\|_p, \|g\|_q>0$. Dividing both sides of $(*)$ by $\|f\|_p\cdot\|g\|_q$, i.e. replacing f,g with $f/\|f\|_p$ and $g/\|g\|_p$, reduces $(*)$ to

$$\|fg\|_1 \le 1$$

for normal f,g, i.e. $\|f\|_p=1$ and $\|g\|_q=1$. To this end, applying the geometric-algebraic averages inequality to $\alpha := \frac{1}{p}$ and $1-\alpha=\frac{1}{q}$, we get:

$|fg| = (|f|^p)^{\frac{1}{p}}\cdot(|g|^q)^{\frac{1}{q}} \le \frac{1}{p}|f|^p + \frac{1}{q}|g|^q$, so integrating gives:

$$\|fg\|_1 \le \frac{1}{p}\int|f|^p\,d\mu + \frac{1}{q}\int|g|^q\,d\mu = \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q = \frac{1}{p}+\frac{1}{q} = 1.$$

$\square$